Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 5, pp. 641–648, 2007

PARTIALLY INVARIANT SOLUTIONS FOR A SUBMODEL OF RADIAL MOTIONS OF A GAS

S. V. Khabirov

UDC 533

All partially invariant solutions in terms of the group of extensions for a model of radial motions of an ideal gas are found. The solutions are obtained by the method of separation of variables in an equation containing functions of one variable but different functions of different independent variables. The solutions predict different continuous unsteady convergence or expansion of the gas under the action of a piston with a point sink or source. If the sink or source affects all particles simultaneously, a collapse or an explosion occurs.

Key words: *radial motion of the gas, collapse, source, separation of variables.*

Introduction. The group analysis of equations of ideal gas dynamics yields a set of invariant submodels [1, 2]. Studying partially invariant submodels involves significant difficulties in studying compatibility of overdetermined systems [3, 4]. For invariant submodels of rank 2, partially invariant submodels of rank 1 and defect 1 reduce to one equation for functions of one variable, but these functions depend on different variables. Separation of variables in this equation is a nontrivial problem, but it can be solved. Examples of such separation for a submodel of radiation motions of a gas are given in the present paper.

1. Formulation of the Problem. The radially symmetric motions of the gas are defined by the equations

$$\begin{gathered} U_t + UU_r + \rho^{-1} p_r = 0, \\ \rho_t + U\rho_r + \rho(U_r + \nu r^{-1} U) = 0, \\ S_t + US_r = 0, \qquad p = f(\rho, S), \end{gathered} \tag{1.1}$$

where U is the radial velocity, ρ is the density, S is the entropy, p is the pressure, $p = f(\rho, S)$ is the equation of state, and $\nu = 0$, 1, or 2 is a parameter of plane, cylindrical, or spherical symmetry of motion, respectively.

Equations (1.1) admit extensions $t\,\partial_t + r\,\partial_r$ [1]. With the use of the invariants $s = rt^{-1}$, U, ρ, and S, one can specify three different presentations of the partially invariant solution of rank 1 and defect 1. In these presentations, one functions has to be of a general form and depend on t and r. This function is called the extra function [3]. The remaining functions depend on the variable s. System (1.1) can be written for functions depending on one variable (s or t) or on the sum of such variables. Hence, the problem of separation of variables arises.

2. Density as an Extra Function. We assume that the density $\rho = \rho(t, s)$ is a function of the general form, and the velocity $U = U(s)$ and entropy $S = S(s)$ are functions of one invariant variable s. Substituting the presentation of the solution into system (1.1), we obtain

$$\begin{gathered} (U - s)S' = 0, \qquad (U - s)U' + \rho^{-1}(f_\rho \rho_s + f_S S') = 0, \\ t\rho_t + (U - s)\rho_s = -\rho(U' + \nu s^{-1} U). \end{gathered} \tag{2.1}$$

For $U = s$, system (2.1) yields a solution $\rho = t^{-\nu-1}\rho_0\mu(S)$, where $S = S(s)$ is an arbitrary function and $\rho = g(p)\mu(S)$ is the equation of state with separated density. Such an isentropic solution was studied in [5].

Institute of Mechanics, Ufa Research Center, Russian Academy of Sciences, Ufa 450054; habirov@anrb.ru.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 5, pp. 26–34, September–October, 2007. Original article submitted April 18, 2006; revision submitted September 21, 2006.

The motion is isentropic ($S = S_0$ for $U \neq s$); Eqs. (2.1) have the integrals

$$U = \varphi'(s), \qquad (\varphi')^2/2 - s\varphi' + \varphi + i(\rho) = D(t), \qquad i = \int \rho^{-1} f_\rho \, d\rho; \tag{2.2}$$

$$\tau'(i)\Big(tD' - (\varphi' - s)^2\varphi''\Big) + \varphi'' + \nu s^{-1}\varphi' = 0, \qquad \ln \rho = \tau(i). \tag{2.3}$$

If we find i from Eqs. (2.2) and substitute it into Eqs. (2.3), we obtain an equality containing functions of one variable $D(t)$, $\varphi(s)$, and $\tau(i)$. It is necessary to separate the variables. We differentiate Eqs. (2.3) with respect to t and s and find the ratio of the resultant equalities:

$$t(D')^{-1}D''(U-s)U'(U' - \nu s^{-1}U) + (U-s)^2\nu s^{-1}UU'' + tD'(U'' + \nu s^{-1}U' - \nu s^{-2}U)$$

$$+ (U-s)U'[2(U')^2 + U'(\nu s^{-1}U + \nu - 1) + \nu s^{-1}U(s^{-1}U - 2)] = 0.$$

Differentiating this expression with respect to t, we obtain an equality in which we can separate the variables s and t under the conditions $(tD')' \neq 0$ and $U' + \nu s^{-1}U \neq 0$:

$$\frac{(t(D')^{-1}D'')'}{(tD')'} = -\frac{U'' + \nu s^{-1}U' - \nu s^{-2}U}{(U-s)U'(U' + \nu s^{-1}U)} = k \qquad (k = \text{const}). \tag{2.4}$$

Integration of Eq. (2.4) leads to the following results for D:

(a) $D = C_0^{-1}(n-1)^{-1}|t|^{1-n}$ for $k = 0$ and $n \neq 1$;

(b) $D = -k^{-1}\ln|C_0 + k(1-n)^{-1}|t|^{1-n}|$ for $n \neq 1$ and $k \neq 0$;

(c) $D = -k^{-1}\ln|C_0 + k\ln|t||$ for $k \neq 0$ and $n = 1$, where C_0 and n are constant.

In case (a), we have $tD' = (1-n)D$. Equation (2.3) acquires the form

$$\tau'(i)\Big[(1-n)(i + (\varphi')^2/2 - s\varphi' + \varphi) - \varphi''(\varphi' - s)^2\Big] + \varphi'' + \nu s^{-1}\varphi' = 0. \tag{2.5}$$

After differentiation with respect to i, the variables are separated:

$$i + \tau'(\tau'')^{-1} = -(\varphi')^2/2 + s\varphi' - \varphi + (1-n)^{-1}\varphi''(\varphi' - s)^2 = A \qquad (A = \text{const}).$$

Integrating this equation, we obtain

$$\tau = (\gamma - 1)^{-1}\ln|C^{-1}(i - A)|, \qquad p = f(\rho) = C_0\rho^\gamma + p_0, \qquad i = A + \gamma C_0(\gamma - 1)^{-1}\rho^{\gamma - 1}.$$

Here γ, p_0, and C_0 are constants of integration.

Substituting the resultant expressions into Eq. (2.5) and setting the coefficient at i to zero, we obtain two equations, one of them determining the function φ by integration. Substituting the same expressions into the second equation and setting the coefficients at different powers of s to zero, we obtain the solution

$$U = 2(\gamma(\nu + 1) + 1 - \nu)^{-1}s, \qquad \gamma \neq 1,$$

$$\frac{\gamma C_0}{\gamma - 1}\rho^{\gamma - 1} = \frac{(\nu+1)(\gamma-1) + 2}{2C_0(\gamma-1)(\nu+1)}|t|^{-2(\gamma-1)(\nu+1)/[(\gamma-1)(\nu+1)+2]} + \frac{(\gamma-1)(\nu+1)s^2}{2(\gamma\nu + \gamma + 1 - \nu)} \tag{2.6}$$

for $\nu \neq 0$ and $n = [3(\gamma-1)(\nu+1)+2]/[(\gamma-1)(\nu+1)+2] \neq 1$ and the solution

$$U = 2(\gamma+1)^{-1}s + U_0,$$

$$\frac{\gamma C_0}{\gamma - 1}\rho^{\gamma-1} = \frac{\gamma + 1}{2C_0(\gamma - 1)}|t|^{2(1-\gamma)/(\gamma+1)} + \frac{\gamma - 1}{(\gamma+1)^2}s^2 - \frac{2U_0 s}{\gamma + 1} + \frac{U_0^2}{\gamma - 1} \tag{2.7}$$

for $\nu = 0$ (U_0 is a constant). The solutions (2.6) and (2.7) have a linear field of velocities and are contained in the solutions obtained in [6, § 16] in the form of series. In the present work, the final formulas for density are obtained.

In case (b), we have $tD' = (n-1)k^{-1}(1 - C_0\exp(kD))$, $n \neq 1$, and $k \neq 0$. Equation (2.3) takes the form

$$(n-1)\Big(1 - C_0\exp(ki)\exp[k((\varphi')^2/2 - s\varphi' + \varphi)]\Big)/k$$

$$-(\varphi' - s)^2\varphi'' + \tau'(i)^{-1}(\varphi'' + \nu s^{-1}\varphi') = 0. \tag{2.8}$$

After differentiation with respect to i, the variables are separated:

$$\frac{(1-n)C_0 \exp[k((\varphi')^2/2 - s\varphi' + \varphi)]}{\varphi'' + \nu s^{-1}\varphi'} = \frac{\tau''}{(\tau')^2}\exp(-ki) = -kA \qquad (kA = \text{const}).$$

This yields $(\tau')^{-1} = A\exp(ki) + B$ (B is a constant of integration). Substituting the latter expression into Eq. (2.8) and setting the coefficient at $\exp(ki)$ to zero for $A \neq 0$, we obtain two equalities for the function $\varphi(s)$:

$$\varphi'' = \frac{(n-1)k^{-1} + B\nu s^{-1}\varphi'}{(\varphi' - s)^2 - B},$$

$$\ln((n-1)A^{-1}k^{-1}C_0) + k((\varphi')^2/2 - s\varphi' + \varphi)' = \ln((n-1)k^{-1} + \nu s^{-1}\varphi'(\varphi' - s)^2) - \ln((\varphi' - s)^2 - B).$$

If $A = 0$, then $(tD')' = 0$ (this case is considered below).

Differentiating the latter equality with respect to s and introducing the variables W and λ instead of U and s through the formulas

$$W = sU^{-1}, \qquad \lambda = (s^2(W^{-1} - 1)^2 - B)((n-1)k^{-1}W + \nu B)^{-1},$$

we obtain the equality

$$\nu\lambda^2((n-2)W + \nu Bk + 1) + \lambda((\nu - 3 + n)W + \nu(Bk - 1)) + 2 = 0, \tag{2.9}$$

which is linear with respect to W, and the equality for the differentials

$$W\Big(\frac{n-1}{k}W + \nu B\Big)\frac{d\lambda}{dW} = 2\Big[B + \lambda\Big(\frac{n-1}{k}W + \nu B\Big)\Big]\Big(\frac{\lambda}{\lambda - 1} + \frac{1}{W-1}\Big) - \frac{n-1}{k}\lambda W.$$

Eliminating W from this equation by virtue of Eq. (2.9), we obtain a relation, which is polynomial with respect to λ. Hence, either $\lambda = \lambda_0$ is a constant, or the resultant relation is an identity with respect to λ. For $\lambda = \lambda_0$, we have $U = s$ (i.e., the case considered above). The identity with respect to λ is written as

$$\nu Bk + (n-1)\Big(\nu(n-2)\frac{\nu(\nu Bk+1)\lambda^2 + \nu(Bk-1)\lambda + 2}{(\nu(n-2)\lambda + n + \nu - 3)^2} - \frac{2\nu(\nu Bk+1)\lambda + \nu(Bk-1)}{\nu(n-2)\lambda + n + \nu - 3}\Big)$$

$$= 2\Big(kB(\nu\lambda + 1) - (n-1)\frac{\nu(\nu Bk+1)\lambda^2 + \nu(Bk-1)\lambda + 2}{\nu(n-2)\lambda + n + \nu - 3}\Big)$$

$$\times\Big(\frac{1}{\lambda - 1} - \frac{\nu(n-2)\lambda + n + \nu - 3}{\nu\lambda^2(\nu Bk + n - 1) + \lambda(\nu Bk + n - 3) + 2}\Big)$$

$$\times\Big(\lambda\frac{2\nu(\nu Bk+1)\lambda + \nu(Bk-1)}{\nu(\nu Bk+1)\lambda^2 + \nu(Bk-1)\lambda + 2} - \frac{\nu(n-2)\lambda}{\nu(n-2)\lambda + n + \nu - 3} - 1\Big).$$

This equation yields $2\nu(n-2)(n+\nu-3)^{-1} = Bk(n + 2\nu - 3) - 2n - 3\nu + 2$ for $\lambda = 0$ and $\nu Bk = 1 - n$ as $\lambda \to \infty$. Hence, $\nu \neq 0$, $n \neq 1$, and $k \neq 0$.

Thus, the equation $n^2 + 4 = 0$ has no real roots for $\nu = 1$ and has the root $n = 1$ for $\nu = 2$. A contradiction arises. Hence, there are no new solutions in case (b).

In case (c), we have $tD' = -\exp(kD)$ and $k \neq 0$. Equation (2.3) acquires the form

$$-\exp(ki)\exp[k((\varphi')^2/2 - s\varphi' + \varphi)] + (\tau')^{-1}(\varphi'' + \nu s^{-1}\varphi') = 0. \tag{2.10}$$

Differentiating this equation with respect to i, we obtain the equality $(\tau')^{-1} = A\exp(ki) + B$, $A \neq 0$. Substituting τ' into (2.10) and setting the coefficient at $\exp(ki)$ to zero, we obtain

$$\varphi'' = \frac{\nu Bs^{-1}\varphi'}{(\varphi' - s)^2 - B}, \qquad \frac{A\nu s^{-1}\varphi'(\varphi' - s)^2}{(\varphi' - s)^2 - B} = \exp[k((\varphi')^2/2 - s\varphi' + \varphi)].$$

Differentiating this expression with respect to s, we obtain relations, which become the linear equation for V after the substitution $U = \varphi' = sV$, $\mu = s^2(V-1)^2 - B$ as

$$V = \frac{\mu(\mu - (\nu - 2)B)}{(\nu B + 1)\mu^2 + \nu B(Bk - 1)\mu + 2\nu B^2}$$

and the differential equation

$$\frac{d\mu}{2(\mu+B)}+\Big(\frac{\mu}{\mu-\nu B}-\frac{V}{V-1}\Big)\,d\ln V=0.$$

This implies the identity with respect to the variable μ:

$$-\frac{1}{2}=\Big(\frac{\mu+B}{\mu-\nu B}+\frac{(\mu-(\nu-2)B)(\mu+B)}{\nu Bk\mu^2+B\mu(\nu Bk-2)+2\nu B^2}\Big)$$

$$\times\Big(\frac{2\mu-(\nu-2)B}{\mu-(\nu-2)B}-\mu\,\frac{2(\nu Bk+1)\mu+\nu B(Bk-1)}{(\nu Bk+1)\mu^2+\nu B(Bk-1)\mu+2\nu B^2}\Big).$$

As $\mu\to\infty$, we have $\nu B=0$; hence, $U=U_0$, i.e., we obtain a trivial solution corresponding to the state at rest. The case $U'+2s^{-1}U=0$ reduces to the case $(tD')'=0$.

We still have to consider the case $tD'=ka_0^2$ (ka_0^2 is a constant). It follows that $D=a_0^2\ln|t|^k$ ($k\neq 0$), and Eq. (2.3) takes the form

$$\tau'(i)[a_0^2k-(\varphi'-s)^2\varphi'']+\varphi''+\nu s^{-1}\varphi'=0. \tag{2.11}$$

After separating the variables i and s, by virtue of Eqs. (2.2), we obtain

$$\tau=a_0^{-2}i+b_0,\qquad p=f(\rho,S_0)=a_0^2\rho+p_0,\qquad i=a_0^2\ln(\rho/\rho_0).$$

In the new variables $U=s+a_0u$ and $s=a_0x$, Eq. (2.11) reduces to the Abel equation

$$\frac{dx}{du}=\frac{x(u^2-1)}{\nu u-x(u^2-k-\nu-1)}, \tag{2.12}$$

while the density is determined by the equality

$$\rho=\rho_0|t|^k\exp\Big(-\frac{1}{2}\,u^2-\int u\,dx\Big). \tag{2.13}$$

Submodel (2.12) for equations of gas dynamics does not seem to be published previously.

For $\nu=0$, Eq. (2.12) has the integral

$$x-x_0=\begin{cases}-u+1/u, & k=-1,\\ -u-(\beta^2+1)\arctan(u/\beta), & k+1=-\beta^2,\\ -u-\dfrac{\alpha^2-1}{2\alpha}\ln\Big|\dfrac{u+\alpha}{u-\alpha}\Big|, & k+1=\alpha^2.\end{cases}$$

For $\nu\neq 0$, Eq. (2.12) has the integral straight line $x=0$ and is invariant during the inversion $x\to -x$, $u\to -u$. The singular point $x=0$, $u=0$ is a saddle. The saddle separatrices have the tangential lines $x=0$ and $(k+\nu+1)x+(\nu+1)a=0$ at the singular point.

If $k+\nu=0$, there are no other singular points. The integral curves for this case are constructed below.

3. Velocity as an Extra Function. Let the velocity $U=U(t,s)$ be a function of the general form and the density $\rho=\rho(s)$ and entropy $S=S(s)$ be functions of one invariant variable $s=rt^{-1}$. It follows from system (1.1) that the gas motion is isentropic ($S=S_0$):

$$U=s^{-\nu}\rho^{-1}\Big(u(t)+\int s^{\nu+1}\rho'(s)\,ds\Big),\qquad u'\neq 0,\qquad i=\int\rho^{-1}f_\rho\rho'\,ds,$$
$$tu'+\Big(u+\int s^{\nu+1}\rho'\,ds-s^{\nu+1}\rho\Big)\Big[s\rho^{-1}\rho'-s^{-\nu}\rho^{-1}(\nu s^{-1}+\rho^{-1}\rho')\Big(u+\int s^{\nu+1}\rho'\,ds\Big)\Big]+s^\nu\rho i'=0. \tag{3.1}$$

Differentiating (3.1) with respect to t, we obtain the equality where the variables t and s can be separated:

$$\frac{1}{u'}\Big(\frac{(tu')'}{u'}\Big)'=\frac{2}{s^\nu\rho}\Big(\frac{\nu}{s}+\frac{\rho'}{\rho}\Big)=2k$$

(k is a constant). Thus, we find $tu'=ku^2+nu+c$ and $s^\nu\rho(R_0+ks)=1$, where n, c, and R_0 are constants of integration. Substituting these expressions into (3.1) and setting the coefficients at the powers of the variable

u to zero, we obtain the equalities $k = 0$, $\rho = \rho_0 s^{-\nu}$, $n = \nu$ ($\nu = 1, 2$), $i = i_0 - c\rho_0^{-1}s - \nu(\nu+1)s^2/2$, and $U = -\nu s + \rho_0^{-1}\nu^{-1}(c_1 t^\nu - c)$, where i_0, c_1, and ρ_0 are constants.

The solution for $\nu = 1$ is

$$\rho = \rho_0 s^{-1}, \qquad U = -s + \rho_0^{-1}(c_1 t - c), \qquad p = p_0 + c\ln\rho - 2\rho_0^2\rho^{-1}, \tag{3.2}$$

and the solution for $\nu = 2$ is

$$\rho = \rho_0 s^{-2}, \qquad U = -2s + \rho_0^{-1}(c_1 t^2 + c)/2, \qquad p = p_0 + c\rho_0^{-1/2}\rho^{1/2} + 3\rho_0 \ln\rho. \tag{3.3}$$

For $c > 0$, the equations of state imply that $p > 0$ in the domain $\rho > \rho_1 > 0$, where $p_\rho > 0$ and $p_{\rho\rho} < 0$. For $c < 0$, the pressure is positive in the domain $0 < \rho_1 < \rho < \rho_2$; $p_\rho > 0$ in the interval $\rho_1 < \rho < \rho_m < \rho_2$, where $p_{\rho\rho} < 0$.

Thus, the equations of state in (3.2) and (3.3) do not satisfy the properties of the standard gas [7, § 2], but the solutions with a linear field of velocities are original.

4. Entropy as an Extra Function. Let the entropy $S = S(t, s)$ be a function of the general form, and the density $\rho = \rho(s)$ and velocity $U = U(s)$ be functions of one invariant variable s. System (1.1) yields the equalities

$$S = t\exp\Big(-\int (U - s)^{-1}\,ds\Big), \qquad p = D(t) - \int \rho(U - s)U'\,ds,$$
$$(U - s)\,d\rho + \rho(dU + \nu s^{-1}U\,ds) = 0. \tag{4.1}$$

Here S is a function of entropy and $D(t)$ is an arbitrary function. Thus, we obtain the equation of state in the form

$$p = f(\rho, S) = D(S\alpha(\rho)) + \beta(\rho),$$

where

$$\alpha(\rho) = \exp\Big(\int (U - s)^{-1}\,ds\Big), \qquad \beta(\rho) = -\int \rho(U - s)U'\,ds.$$

The function $U(s)$ being defined, we can determine the density $\rho(s)$ from Eqs. (4.1) and, hence, the functions $\alpha(\rho)$ and $\beta(\rho)$. Thus, there exists an equation of state at which the gas motion proceeds with the velocity $U(s)$.

If $\alpha(\rho)$ and $\beta(\rho)$ are prescribed, the following relations are valid:

$$\beta' = (V - 1)^2 s^2(\nu V\rho\alpha^{-1}\alpha' + 1), \qquad U = sV,$$
$$(V - 1)\alpha^{-1}\alpha'\,d\rho = s^{-1}\,ds = -\Big((\nu + 1)V + \alpha\rho^{-1}(\alpha')^{-1}\Big)^{-1}\,dV. \tag{4.2}$$

Eliminating the variable s from here, we obtain the differential equation (4.2) and the final relation between the quantities ρ and V:

$$V^2(3\nu + 1)\rho\alpha^{-2}(\alpha')^2 + V\Big(\nu\rho\alpha'\alpha^{-1}\beta''(\beta')^{-1} + 4\nu\alpha^{-1}\alpha' - \nu\rho\alpha^{-1}\alpha''$$
$$- \nu(\nu - 2)\rho\alpha^{-2}(\alpha')^2\Big) + (\beta')^{-1}\beta'' + 2\rho^{-1} + (2 - \nu)\alpha^{-1}\alpha' = 0. \tag{4.3}$$

Relation (4.3) is not an identity with respect to the variable V; instead, it defines the function $V(\rho)$. If this function is substituted into Eqs. (4.2), we obtain a differential relation between α and β. Thus, only one function (α or β) in the equation of state can be chosen arbitrarily. New exact solutions depending on the form of the equation of state from the family with single-functional arbitrariness are obtained.

5. Gas Motion in the Case of Exact Solutions. Formulas (2.6) define an isentropic flow of a polytropic gas with the world lines

$$r = C|t|^k, \qquad k = 2(\gamma(\nu + 1) + 1 - \nu)^{-1} \neq 1, \qquad \nu = 1, 2,$$

where γ is the ratio of specific heats $p = C_0\rho^\gamma + p_0$ and C is the Lagrangian coordinate of particles. If $1 < \gamma < 2$, then $1/2 < k < 1$ for $\nu = 1$ and $2/5 < k < 1$ for $\nu = 2$. If $\gamma > 1$, then $k < 1$. As $t \to -0$, the particles converge toward the center with infinite velocities, and the density tends to infinity (Fig. 1). For $-\infty < t < 0$, a collapse into a point occurs; for $0 < t < \infty$, the particles move away from the center with infinite velocities [6, § 16].

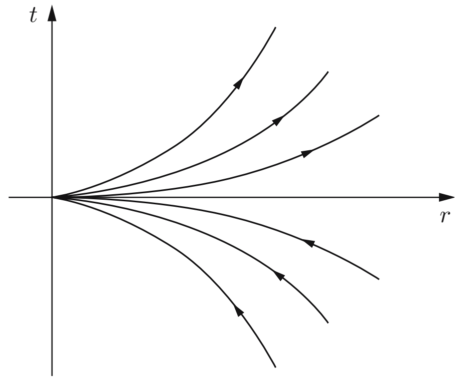

Fig. 1. Collapse and spreading of particles.

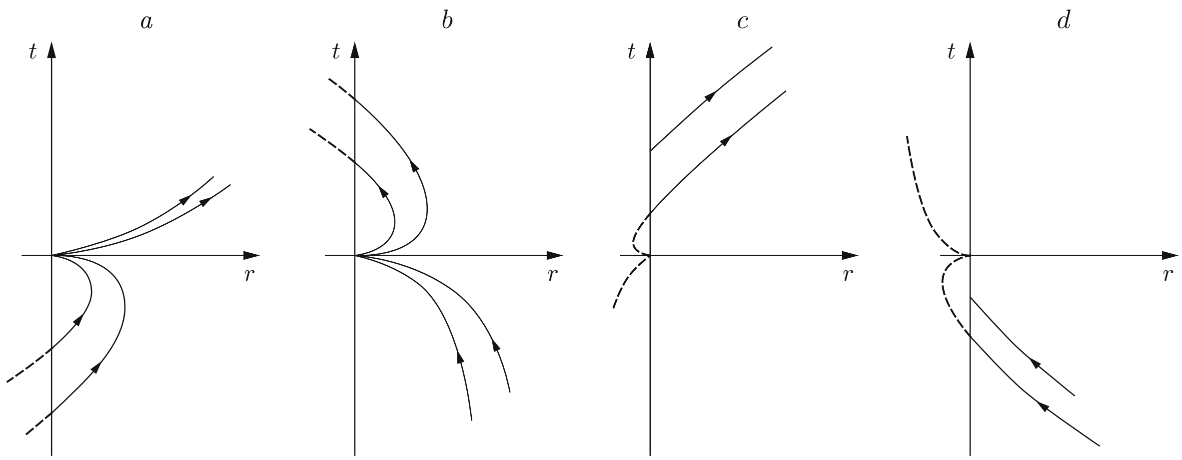

Fig. 2. Four types of motion with plane waves: (a) source with a collapse and a point explosion; (b) collapse and a point explosion with a sink; (c) source; (d) sink.

Formulas (2.7) define an isentropic flow with plane waves ($\nu = 0$) of a polytropic gas with the world lines

$$r = C|t|^{2/(\gamma+1)} + U_0 \frac{\gamma+1}{\gamma-1} t,$$

where C is the Lagrangian coordinate of particles. For $t = t_c$, where $|t_c| = [(C/U_0)(\gamma-1)/(\gamma+1)]^{(\gamma+1)/(\gamma-1)}$, the particle is located in the origin of the coordinate system ($r = 0$) and has a velocity $(dr/dt)|_{t=t_c} = U_0$. Four types of motion are possible: 1) source with a collapse and a point explosion for $C > 0$ and $U_0 > 0$ (Fig. 2a); 2) collapse and a point explosion with a sink for $C > 0$ and $U_0 < 0$ (Fig. 2b); 3) source for $C < 0$ and $U_0 > 0$ (Fig. 2c); 4) sink for $C < 0$ and $U_0 < 0$ (Fig. 2d).

Let us consider the Abel equation (2.12) with one singular point $(0, 0)$ (saddle) for $\nu + k = 0$. The saddle separatrix has the following asymptotics:

$$x \sim -(\nu+1)u - \frac{\nu(\nu+1)}{\nu+2} u^3 + O(u^5) \qquad \text{as} \qquad u \to -0,$$

$$u \sim -1 + \frac{\nu}{2x} - \frac{\nu^2}{8x^2} + \frac{\nu^2}{4x^3} + o(x^{-3}) \qquad \text{as} \qquad x \to \infty,\ u \to -1.$$

Let us calculate the limits characterizing the behavior of the integral curves in the upper half-plane $x > 0$ (Fig. 3):

$$\frac{dx}{du}\Big|_{u=\pm 1} = 0, \qquad \frac{d^2x}{du^2}\Big|_{u=\pm 1} = \frac{2x}{\nu}$$

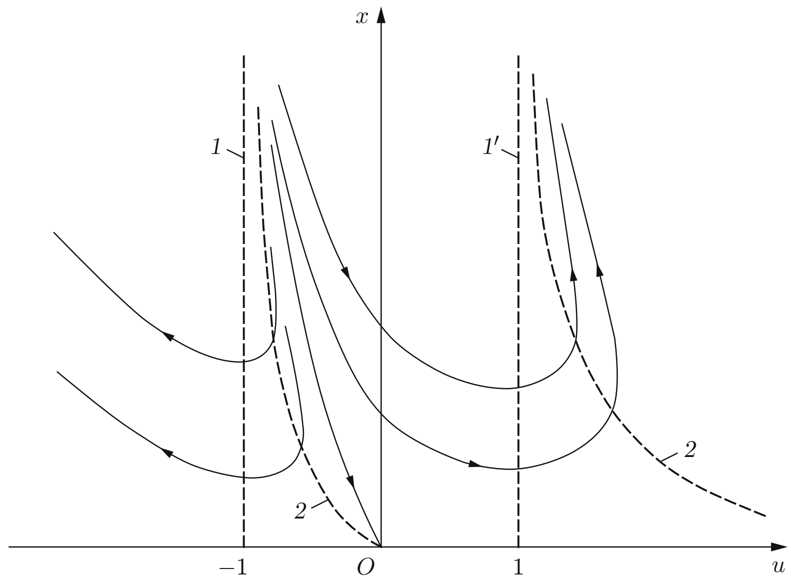

Fig. 3. Integral curves of the Abel equation with one singular point: $u = -1$ (1), $u = 1$ (1′), and $x(u^2 - 1) = \nu u$ (2).

(hence, there are minimums on the straight lines $u = \pm 1$) and

$$\frac{dx}{du}\Big|_{u=0} = -1, \qquad \frac{du}{dx} = 0, \qquad \frac{d^2u}{dx^2} = -\frac{1}{x} < 0$$

[hence, there are maximums on the curve $x(u^2 - 1) = \nu u$].

For all integral curves, we have

$$u - \frac{1}{u} = \frac{\nu}{x}\Big(1 + \frac{\nu}{2x^2} + O(x^{-4})\Big) > 0 \qquad \text{as} \qquad x \to \infty,\ u \to \pm 1,$$

hence,

$$u \sim \frac{\nu}{2x}\Big(1 + \frac{\nu}{2x^2}\Big) \pm \Big(1 + \frac{\nu^2}{8x^2}\Big).$$

As the curve with the maximums has the asymptotics $u = \pm 1$, all integral curves asymptotically approach this curve as $x \to \infty$ and $u \to \pm 1$.

Along all straight lines $x = lu$, $l \neq 0$, we have

$$\frac{dx}{du}\Big|_{x=lu} = \frac{l(u^2 - 1)}{\nu - l(u^2 - 1)} \to -1 \qquad \text{as} \qquad u \to \infty.$$

As $u \to -\infty$, the integral curves have the asymptotics $x = C - u - \nu u^{-1} - C\nu u^{-2}/2 + \dots$.

For all integral curves of Eq. (2.12) in Fig. 3, the world lines of particles are determined from the equation

$$\frac{dr}{dt} = U = a_0(x + u), \qquad r = a_0 t x.$$

Hence, $dx/u = dt/t$, and the density along the world line is determined by equality (2.13).

As $u \to -0$ (point O in Fig. 3), along the saddle separatrix we have $x \sim -(\nu + 1)u$, $t \sim C|u|^{-\nu-1}$, and $r \sim Ca_0(\nu + 1)|u|^{-\nu}$, where C defines the particle. Thus, $t \to \infty$, $r \to \infty$, and $dr/dt \sim -\nu a_0 u \to 0$, and the saddle corresponds to spreading of particles with zero velocities into vacuum ($\rho \to 0$ at infinity).

Along the integral curves asymptotically approaching the asymptote $u = -1$ (Fig. 3), we have

$$u \to -1 + 0, \quad x \sim \nu(u+1)^{-1}/2, \quad t \sim C|1 + u^{-1}|^{\nu/2} \exp\left(-\nu(u+1)^{-1}/2\right) \to 0,$$

$$r \sim (1/2)\nu a_0 C|u|^{-1}|1 + u^{-1}|^{-1+\nu/2} \exp\left(-\nu(u+1)^{-1}/2\right) \to 0, \quad \frac{dr}{dt} \sim \nu a_0 (u+1)^{-1}/2 \to \infty.$$

As $t \to -0$, such a behavior of the solution corresponds to a collapse of particles at the point $r = 0$ with infinite velocities, and $\rho \to \infty$.

Along the integral curves asymptotically approaching the asymptote $u = 1$ (Fig. 3), we have

$$u \to 1 + 0, \quad x \sim \nu(u-1)^{-1}/2, \quad t \sim C|1 - u^{-1}|^{\nu/2} \exp\left(\nu(u-1)^{-1}/2\right) \to \infty,$$

$$r \sim (1/2)\nu a_0 C u^{-1}|1 - u^{-1}|^{-1+\nu/2} \exp\left(\nu(u-1)^{-1}/2\right) \to \infty, \quad \frac{dr}{dt} \sim a_0(u + \nu(u-1)^{-1}/2) \to \infty.$$

Such a behavior of the solution corresponds to spreading of particles with infinite velocities into vacuum ($\rho \to 0$ at infinity).

Along the integral curve with the asymptote $x = C - u$ (see Fig. 3), we have

$$u \to -\infty, \quad x \sim C - u - \nu u^{-1}, \quad t \sim C_1|u|^{-1} \exp\left(\nu(u-1)^{-1}/2\right) \to 0,$$

$$r \sim a_0 C_1 |u|^{-1} \exp\left(\nu u^{-2}/2\right)(C - u - \nu u^{-1}) \to a_0 C_1, \quad \frac{dr}{dt} \sim a_0(C - \nu u^{-1}) \to a_0 C.$$

Such a behavior of the solution corresponds to spreading of particles with finite velocities from spheres of radius $a_0 C_1$ into a medium with the density distribution $\rho = \rho_0 a_0^\nu r^{-\nu}$ at the initial time.

The pattern of the world lines is qualitatively consistent with Fig. 1.

Remark 1. Solutions (3.2) and (3.3) for a gas with equations of state that do not satisfy the properties of the standard gas define continuous convergence and expansion of the gas with a possible sink and source at the center, where the density and pressure acquire infinite values. The world line that does not pass through the center can be considered as a piston.

This work was supported by the Russian Foundation for Basic Research (Grant Nos. 05-01-000080-a and 05-01-00775-a) and Program of the President of the Russian Federation on the State Support of Leading Scientific Schools (Grant No. NSh-5245.2006.1).

REFERENCES

1. L. V. Ovsyannikov, "SUBMODELS program. Gas dynamics," *Prikl. Mat. Mekh.*, **58**, No. 4, 30–55 (1994).
2. S. V. Khabirov, "Invariant solutions of equations of gas dynamics," *Vestn. Ufim. Gos. Aviats.-Tekh. Inst.*, No. 1, 47–52 (2001).
3. L. V. Ovsyannikov, "Summarizing of SUBMODELS application to gas-dynamic equations," *Prikl. Mat. Mekh.*, **63**, No. 3, 362–372 (1999).
4. S. V. Khabirov, "Irregular, partially invariant solutions of rank 2 and defect 1 of gas-dynamics equations," *Sib. Mat. Zh.*, **43**, No. 5, 1168–1181 (2002).
5. S. V. Khabirov, "Continuous restricted radial motion of a gas under the action of a piston," *J. Appl. Mech. Tech. Phys.*, **45**, No. 2, 249–259 (2004).
6. L. I. Sedov, *Similarity and Dimensional Analysis*, Academic Press, New York (1959).
7. L. V. Ovsyannikov, *Lectures on Gas Dynamics* [in Russian], Nauka, Moscow (1981).